A RECIPROCITY THEOREM FOR UNITARY REPRESENTATIONS OF LIE GROUPS

BY

HENRI MOSCOVICI

ABSTRACT

Let G be a Lie group, H a closed subgroup, L a unitary representation of H and U^L the corresponding induced representation on G. The main result of this paper, extending Ol'šanskiĭ's version of the Frobenius reciprocity theorem, expresses the intertwining number of U^L and an irreducible unitary representation V of G in terms of L and the restriction of V_∞ to H.

1.

Throughout, G will be a Lie group and H a closed subgroup. For any strongly continuous representation R we denote by $\boldsymbol{H}(R)$ the Hilbert space on which R acts and by $\boldsymbol{H}_\infty(R)$ the linear subspace of all C^∞-vectors for R, topologized as in [3]; $\boldsymbol{H}_\infty(R)$ is then a Fréchet space and, corresponding to R, there are natural representations R_∞ and R_∞^* whose representation spaces are $\boldsymbol{H}_\infty(R)$ and its weak dual $\boldsymbol{H}_\infty(R)^*$, respectively. If δ_G and δ_H are the modular functions of G and H, we set $\delta(h) = \delta_G(h)^{-\frac{1}{2}}\delta_H(h)^{\frac{1}{2}}$, $h \in H$.

Let U^L be a unitary representation of G induced by a unitary representation L of H (see [1] for the definition) and let V be an arbitrary irreducible unitary representation of G. Assuming G/H compact and L finite-dimensional, Ol'šanskiĭ [6] proved that the intertwining spaces $\mathrm{Hom}_G(\boldsymbol{H}(V), \boldsymbol{H}(U^L))$ and $\mathrm{Hom}_H(\boldsymbol{H}(\delta \otimes L)^*, \boldsymbol{H}_\infty(V)^*)$ are naturally isomorphic. But L being finite-dimensional, the second space can be replaced in this statement by $\mathrm{Hom}_H(\boldsymbol{H}_\infty(V), \boldsymbol{H}(\delta \otimes L))$. Our Theorem 2 asserts that, thus reformulated, the result remains valid without assuming L to be of finite dimension. If G/H is not compact it is still possible to give a similar description of $\mathrm{Hom}_G(\boldsymbol{H}(V), \boldsymbol{H}(U^L))$. Specifically, we

Received October 10, 1972

prove that $\mathrm{Hom}_G(\boldsymbol{H}(V),\boldsymbol{H}(U^L))$ is isomorphic to a certain linear subspace of $\mathrm{Hom}_H(\boldsymbol{H}_\infty(V),\boldsymbol{H}(\delta\otimes L))$ (Theorem 1), which, of course, coincides with the whole space when G/H is compact. There is a close analogy between this theorem and the reciprocity theorem of K. Maurin and L. Maurin [4], the only difference being that, in their formulation, the role of $\boldsymbol{H}_\infty(V)$ is assumed by another dense subspace of $\boldsymbol{H}(V)$, denoted here by $\boldsymbol{H}_0(V)$, which is the image of the projective tensor product $C_c(G)\hat{\otimes}\boldsymbol{H}(V)$ under the map $\phi\otimes v\mapsto V(\phi)v$, equipped with the direct image of the topology of $C_c(G)\hat{\otimes}\boldsymbol{H}(V)$. The advantage of working with $\boldsymbol{H}_\infty(V)$ instead of $\boldsymbol{H}_0(V)$ is due, among other things, to the fact that there are nice descriptions of the first space, such as Nelson's and Goodman's characterizations of C^∞-vectors.

The arguments proving Theorem 1 also lead to a criterion for irreducibility of an induced representation (Theorem 3) which turns out to be useful in some cases, although its applicability seems to be rather limited.

All proofs are very straightforward. The main reference is Poulsen's paper [7].

2.

Let L be a unitary representation of H and V a unitary representation of G. We begin with the construction of an injective linear mapping $i:\mathrm{Hom}_G(\boldsymbol{H}_\infty(V),\boldsymbol{H}_\infty(U^L))\to\mathrm{Hom}_H(\boldsymbol{H}_\infty(V),\boldsymbol{H}(\delta\otimes L))$. Let $A_\infty\in\mathrm{Hom}_G(\boldsymbol{H}_\infty(V),\boldsymbol{H}_\infty(U^L))$. Since $\boldsymbol{H}_\infty(U^L)$ is contained in $C^\infty(G,\boldsymbol{H}(L))$ (cf. [7, Th. 5.1]), we may define a mapping $\alpha:\boldsymbol{H}_\infty(V)\to\boldsymbol{H}(L)$ by putting

$$\alpha v=(A_\infty v)(e)\quad\text{for } v\in\boldsymbol{H}_\infty(V),$$

where e is the unity of G. Then α is continuous, since the point evaluation $f\mapsto f(e)$ from $\boldsymbol{H}_\infty(U^L)$ to $\boldsymbol{H}(L)$ is continuous (cf. [7, Corollary 5.1]), and

$$\alpha V(h)v=(A_\infty V(h)v)(e)=(U^L(h)A_\infty v)(e)=(A_\infty v)(h)=\delta(h)L(h)\alpha v,$$

for any $h\in H$, that is $\alpha\in\mathrm{Hom}_H(\boldsymbol{H}_\infty(V),\boldsymbol{H}(\delta\otimes L))$. We set $i(A_\infty)=\alpha$. The injectivity of i easily follows noting that

$$\alpha V(g)v=(A_\infty V(g)v)(e)=(U^L(g)A_\infty v)(e)=(A_\infty v)(g),$$

for any $g\in G$.

We want to characterize the image of i. Let α be arbitrary in $\mathrm{Hom}_H(\boldsymbol{H}_\infty(V),\boldsymbol{H}(\delta\otimes L))$; for any $v\in\boldsymbol{H}_\infty(V)$ we define a mapping $\alpha_v:G\to\boldsymbol{H}(L)$ by

$$\alpha_v(g) = \alpha V(g)v, \qquad g \in G.$$

LEMMA 1. *Let $v \in H_\infty(V)$. Then:*

i) *α_v is of class C^∞;*

ii) *$\alpha_v(hg) = \delta(h)L(h)\alpha_v(g)$, for $g \in G$, $h \in H$;*

iii) *if $\tilde{X}$ denotes the left invariant differential operator on G corresponding to an element X in the universal enveloping algebra $\mathfrak{U}(\mathfrak{G})$ of the Lie algebra $\mathfrak{G}$ of G, we have $\tilde{X}\alpha_v = \alpha_{dV(X)v}$.*

PROOF. The first assertion is a consequence of the fact that $g \mapsto V(g)v$ is a C^∞-mapping from G to $H_\infty(V)$ [7, Proposition 1.2] and the second follows from the very definition of α_v. To prove (iii), it suffices to verify that $\tilde{X}\alpha_v = \alpha_{dV(X\ v}$ for $X \in \mathfrak{G}$. But then

$$(\tilde{X}\alpha_v)(g) = \frac{d}{dt}\alpha_v(g \cdot \exp tX)\Big|_{t=0} = \frac{d}{dt}\,\alpha V(g)V(\exp tX)v\Big|_{t=0} = \alpha V(g)dV(X)v\,.$$

Q.E.D.

REMARK. The properties (i) and (ii) imply in particular that, using the notation in [1], $\alpha_v \in F^*$.

For an adequate formulation of the reciprocity theorem we introduce the following notion which corresponds to Maurin's concept of a (V, L)-automorphic form.

DEFINITION. An intertwining operator $\alpha \in \mathrm{Hom}_H(H_\infty(V), H(\delta \otimes L))$ is called *essential* if for any $v \in H_\infty(V)$, $\alpha_v \in H(U^L)$. The linear subspace of $\mathrm{Hom}_H(H_\infty(V), H(\delta \otimes L))$ consisting of all essential intertwining operators will be denoted by $\mathrm{Hom}^e_H(H_\infty(V), H(\delta \otimes L))$.

Note that if $\alpha = i\,(A_\infty)$ with $A_\infty \in \mathrm{Hom}_G\,(H_\infty\,(V),\ H_\infty\,(U^L))$ then $\alpha_v = A_\infty v \in H(U^L)$, hence α is in $\mathrm{Hom}^e_G(H_\infty(V), H(\delta \otimes L))$.

LEMMA 2. $\mathrm{Hom}^e_H(H_\infty(V), H(\delta \otimes L))$ *coincides with the image of i, so that i: $\mathrm{Hom}_G(H_\infty(V), H_\infty(U^L)) \to \mathrm{Hom}^e_H(H_\infty(V), H(\delta \otimes L))$ is an isomorphism.*

PROOF. For $\alpha \in \mathrm{Hom}^e_H(H_\infty(V), H(\delta \otimes L))$ let $\tilde{\alpha}: H_\infty(V) \to H(U^L)$ be the mapping which carries v into α_v. By Lemma 1 (iii) we know that $\tilde{X}(\tilde{\alpha}(v)) = \tilde{\alpha}(dV(X)v)$ and so $\tilde{X}(\tilde{\alpha}(v)) \in H(U^L)$ for any $X \in \mathfrak{U}(\mathfrak{G})$ and $v \in H_\infty(V)$. By [7, Th. 5.1] this means that $\tilde{\alpha}(v) \in H_\infty(U^L)$. We shall prove now that $\tilde{\alpha}: H_\infty(V) \to H_\infty(U^L)$ is closed and so is continuous. Let $\{v_n\}$ be a sequence in $H_\infty(V)$ such that $v_n \to 0$ and $\tilde{\alpha}(v_n) \to f \in H_\infty(U^L)$; since the point evaluation is continuous on $H_\infty(U^L)$

(cf. [7, Corollary 5.1]), we get $\tilde{\alpha}(v_n)(g) \to f(g)$ for each $g \in G$. But $\tilde{\alpha}(v_n)(g) = \alpha V(g)v_n \to 0$, hence $f=0$. In addition

$$(\tilde{\alpha}(V(g)v))(x) = \alpha V(x)V(g)v = (\tilde{\alpha}(v))(xg) = (U^L(g)\tilde{\alpha}(v))(x), \quad x \in G,$$

hence $\tilde{\alpha}$ intertwines V_∞ and U^L_∞. Clearly $i(\tilde{\alpha}) = \alpha$. Q.E.D

Now let $A \in \mathrm{Hom}_G\,(\boldsymbol{H}(V), \boldsymbol{H}(U^L))$ and let $r(A)$ denote its restriction to $\boldsymbol{H}_\infty(V)$. By [7, Th. 3.1] $r(A) \in \mathrm{Hom}_G(\boldsymbol{H}_\infty(V), \boldsymbol{H}_\infty(U^L))$. Then $j = i \cdot r$ is a linear mapping of $\mathrm{Hom}_G(\boldsymbol{H}(V), \boldsymbol{H}(U^L))$ to $\mathrm{Hom}^e_H(\boldsymbol{H}_\infty(V), \boldsymbol{H}(\delta \otimes L))$.

THEOREM 1. *Let V be an irreducible unitary representation of G and L a unitary representation of H. Then*

$$j: \mathrm{Hom}_G(\boldsymbol{H}(V), \boldsymbol{H}(U^L)) \to \mathrm{Hom}^e_H(\boldsymbol{H}_\infty(V), \boldsymbol{H}(\delta \otimes L))$$

s an isomorphism.

PROOF. Since i is an isomorphism it remains to show that

$$r: \mathrm{Hom}_G(\boldsymbol{H}(V), \boldsymbol{H}(U^L)) \to \mathrm{Hom}_G(\boldsymbol{H}_\infty(V), \boldsymbol{H}_\infty(U^L))$$

is also an isomorphism. Since $\boldsymbol{H}_\infty(V)$ is dense in $\boldsymbol{H}(V)$, r is injective. On the other hand, if $A_\infty \in \mathrm{Hom}_G(\boldsymbol{H}_\infty(V), \boldsymbol{H}_\infty(U^L))$, according to [7, Th. 3.2] A_∞ has a unique closed extension A from $\boldsymbol{H}(V)$ to $\boldsymbol{H}(U^L)$ which intertwines V and U^L. To prove that $A \in \mathrm{Hom}_G(\boldsymbol{H}(V), \boldsymbol{H}(U^L))$, let $A = T|A|$, where $|A| = (A^*A)^{\frac{1}{2}}$, be the polar decomposition; $|A|$ is a self-adjoint operator which commutes with all $V(g)$, $g \in G$ and V is irreducible, hence by [5, §17, no. 6], $|A|$ must be scalar and so A is everywhere defined. Now obviously $r(A) = A_\infty$. Q.E.D

There is an important case in which $\mathrm{Hom}^e_H(\boldsymbol{H}_\infty(V), \boldsymbol{H}(\delta \otimes L))$ coincides with the whole $\mathrm{Hom}_H(\boldsymbol{H}_\infty(V), \boldsymbol{H}(\delta \otimes L))$, namely when G/H is compact.

THEOREM 2. *Let V be an irreducible unitary representation of G and L a unitary rerepsentation of H. Suppose G/H is compact. Then*

$$\mathrm{Hom}_G(\boldsymbol{H}(V), \boldsymbol{H}(U^L)) \cong \mathrm{Hom}_H(\boldsymbol{H}_\infty(V), \boldsymbol{H}(\delta \otimes L)).$$

PROOF. Since G/H is compact we can choose a positive compactly supported function ϕ on G with the property: $\int_H \phi(hg)dh = 1$ for all $g \in G$ (see, e.g., [2, Ch. VII, §2, no. 2]). Now if $\alpha \in \mathrm{Hom}_H(\boldsymbol{H}_\infty(V), \boldsymbol{H}(\delta \otimes L))$ then

$$\| \alpha_v \|^2 = \int_G \phi(g) \| \alpha V(g)v \|^2 dg < \infty,$$

hence $\alpha_v \in \boldsymbol{H}(U^L)$ for any $v \in \boldsymbol{H}_\infty(V)$. Therefore $\mathrm{Hom}_H(\boldsymbol{H}_\infty(V), \boldsymbol{H}(\delta \otimes L)) = \mathrm{Hom}_H^e(\boldsymbol{H}_\infty(V), \boldsymbol{H}(\delta \otimes L))$.

Q.E.D.

REMARK. In the above statement $\mathrm{Hom}_H(\boldsymbol{H}_\infty(V), \boldsymbol{H}(\delta \otimes L))$ can be replaced by $\mathrm{Hom}_H(\boldsymbol{H}_\infty(V), \boldsymbol{H}_\infty(\delta \otimes L))$. Indeed, the inclusion $\mathrm{Hom}_H(\boldsymbol{H}_\infty(V), \boldsymbol{H}_\infty(\delta \otimes L)) \subseteq \mathrm{Hom}_H(\boldsymbol{H}_\infty(V), \boldsymbol{H}(\delta \otimes L))$ is obvious. Conversely, suppose $\alpha \in \mathrm{Hom}_H(\boldsymbol{H}_\infty(V), \boldsymbol{H}(\delta \otimes L))$. For $v \in \boldsymbol{H}_\infty(V)$ the mapping $h \mapsto \delta(h)L(h)\alpha_v = \alpha_v(h)$ is C^∞ by Lemma 1 (i). Thus $\alpha_v \in \boldsymbol{H}_\infty(\delta \otimes L)$ for every $v \in \boldsymbol{H}_\infty(V)$. The continuity of α, viewed as a mapping of $\boldsymbol{H}_\infty(V)$ into $\boldsymbol{H}_\infty(\delta \otimes L)$, is a direct consequence of the closed graph theorem.

3.

The results in the preceding section lead in particular to the following criterion for irreducibility of an induced representation.

THEOREM 3. *Let L be a unitary representation of H. Then U^L is irreducible if and only if* $\mathrm{Hom}_H^e(\boldsymbol{H}_\infty(U^L), \boldsymbol{H}(\delta \otimes L))$ *is one-dimensional.*

PROOF. If U^I is irreducible then by Theorem 1

$$\dim \mathrm{Hom}_H^e(\boldsymbol{H}_\infty(U^L), \boldsymbol{H}(\delta \otimes L)) = \dim \mathrm{Hom}_G(\boldsymbol{H}(U^L), \boldsymbol{H}(U^L)) = 1.$$

Conversely, suppose $\dim \mathrm{Hom}_H^e(\boldsymbol{H}_\infty(U^L), \boldsymbol{H}(\delta \otimes L)) = 1$. Then, by Lemma 2, $\dim \mathrm{Hom}_G(\boldsymbol{H}_\infty(U^L), \boldsymbol{H}_\infty(U^L)) = 1$. On the other hand $\dim \mathrm{Hom}_G(\boldsymbol{H}(U^L), \boldsymbol{H}(U^L)) \leqq \dim \mathrm{Hom}_G(\boldsymbol{H}_\infty(U^L), \boldsymbol{H}_\infty(U^L))$ since the restriction mapping r:

$$\mathrm{Hom}_G(\boldsymbol{H}(U^L), \boldsymbol{H}(U^L)) \to \mathrm{Hom}_G(\boldsymbol{H}_\infty(U^L), \boldsymbol{H}_\infty(U^L))$$

is injective. Thus $\dim \mathrm{Hom}_G(\boldsymbol{H}(U^L), \boldsymbol{H}(U^L)) = 1$. Q.E.D.

COROLLARY. *Let L be a unitary representation of H. If* $\mathrm{Hom}_H(\boldsymbol{H}_\infty(U^L), \boldsymbol{H}(\delta \otimes L))$ *is one-dimensional, then U^L is irreducible.*

We close this section with two simple examples which show how this last result can be handled in concrete situations.

EXAMPLE 1. Let G be the "Heisenberg group" of all real 3×3 matrices of the form

$$g = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrices

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

form a basis for the Lie algebra of G.

It is well-known that all the infinite-dimensional irreducible unitary representations of G are the representations

$$(V_\lambda(g)f)(t) = \exp(i\lambda(bt + c + ab/2))f(t+a),$$

$$g = \exp(aX + bY + cZ), \qquad f \in L^2(\boldsymbol{R})$$

acting on $L^2(\boldsymbol{R})$, where $\lambda \in \boldsymbol{R}$, $\lambda \neq 0$.

The representation V_λ maps X, Y and Z into d/dt, multiplication by $i\lambda t$ and multiplication by $i\lambda$ respectively. Using Goodman's characterization of C^∞-vectors [3, Th. 1.1] it follows that the space of C^∞-vectors for V_λ is the Schwartz space of rapidly decreasing functions on $\boldsymbol{R}$.

Now we show that the irreducibility of V_λ can be easily deduced by applying the above corollary. First we note that V_λ is unitarily equivalent with the representation of G induced by the character

$$L_\lambda(h) = e^{i\lambda c}, \qquad h = \exp(bY + cZ)$$

of the subgroup $H = \{\exp(bY + cZ);\ b, c \in \boldsymbol{R}\}$. Now, an element $\alpha \in \mathrm{Hom}_H(\boldsymbol{H}_\infty(V_\lambda), \boldsymbol{H}(L_\lambda))$ is exactly a tempered distribution with the property

$$\int e^{i\lambda(bt+c)} f(t)d\alpha(t) = e^{i\lambda c}\int f(t)d\alpha(t) \qquad \text{for any } b, c \in \boldsymbol{R}.$$

But then α must be a scalar multiple of the Dirac distribution δ_0. So

$$\dim \mathrm{Hom}_H(\boldsymbol{H}_\infty(V_\lambda), \boldsymbol{H}(L_\lambda)) = 1.$$

EXAMPLE 2. Let G be the group of affine transformations of the real line realized as the group of all 2×2 matrices of the form

$$g = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \quad \text{with } a > 0, \quad b \in \boldsymbol{R}\ .$$

Consider the unitary representations of G on $L^2(\boldsymbol{R})$

$$(V_\pm(g)f)(t) = \exp(\pm iae^t)f(t+b), \qquad g = \exp aX \cdot \exp bY,$$

where

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

is a basis for the Lie algebra of G. The representation $V_{\pm}$ maps X into multiplication by $\pm\, ie^t$ and Y into d/dt. Accordingly, the space of C^∞-vectors for $V_{\pm}$ consists of all C^∞-functions f with the property: $f^{(n)}(t)$ and $e^{nt}f(t)$ are in $L^2(\boldsymbol{R})$ for any integer $n \geqq 0$. On the other hand, $V_{\pm}$ is unitarily equivalent with the representation induced by the character

$$L_{\pm}(h) = e^{\pm ia}\ , \quad h = \exp aX$$

of the subgroup $H = \{\exp aX;\ a \in \boldsymbol{R}\}$.

Now, an element $\alpha \in \mathrm{Hom}_H(\boldsymbol{H}_\infty(V_{\pm}), \boldsymbol{H}(L_{\pm}))$ defines a distribution which satisfies the condition

$$\int \exp(\pm\, iae^t) f(t) d\alpha(t) = e^{\pm ia} \int f(t) d\alpha(t), \qquad \text{for any } a \in \boldsymbol{R}.$$

But again this happens only when α is proportional to the Dirac distribution δ_0. This proves the irreducibility of $V_{\pm}$.

References

1. R. J. Blattner, *On induced representations*, Amer. J. Math. **83** (1961), 79–98.

2. N. Bourbaki, *Intégration*, Chapters VII and VIII, Hermann, Paris, 1963.

3. R. Goodman, *Analytic and entire vectors for representations of Lie groups*, Trans. Amer. Math. Soc. **143** (1969), 55–76.

4. K. Maurin and L. Maurin, *A generalization of the duality theorem of Gelfand and Piatecki-Šapiro and Tamagawa automorphic forms*, J. Fac. Sci. Univ. Tokyo Sect. I, **17** Parts 1 & 2, 331–340.

5. M. A. Naimark, *Normed Rings*, Noordhoff, Groningen, The Netherlands, 1964.

6. G. I. Ol'šanskiĭ, *The Frobenius duality theorem* (Russian), Funkcional. Anal. i Priložen. **3** (1969), 49–58.

7. N. S. Poulsen, *On C^∞-vectors and intertwining bilinear forms for representations of Lie groups*, J. Functional Analysis **9** (1972), 87–120.

Institute of Mathematics of the Academy
Bucharest, Romania